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Self-duality and periodicity at finite filling fraction

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Abstract

We investigate a model of interacting charged particles in two space dimensions, with manifest invariance under duality and periodicity under flux attachment. This model, introduced by Fradkin and Kivelson (1996 *Nucl. Phys.* B **474** 543), shares many qualitative features of real quantum Hall systems. We extend this model to the case of finite filling fraction, i.e., to physical systems without particle–hole symmetry and without time-reversal invariance. We derive the transformation laws for the the average currents and prove that they have an $SL(2, Z)$ symmetry. We can then calculate the filling factors at the modular fixed points and further explore the topological order of the model by constructing the hierarchy of states.

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1. Introduction

The experimental work of Shahar *et al* [1] on the quantum Hall–insulator phase transition has inspired several theoretical investigations on the duality properties of strongly correlated electronic systems. The experiment looks at the transition between the $1/3$ Hall fluid and the proximate insulator state and finds that the nonlinear I–V_{xx} characteristics are symmetric: for every filling factor ν in the Hall state there exists a dual filling factor ν^d in the insulating state such that the current at ν equals the longitudinal electric field (voltage) at ν_d and the other way around. The identity between the I–V_{xx} traces in the FQH liquid and the V_{xx}–I in the insulator was seen as a proof for the existence of a duality symmetry between charge and flux.

The phase diagram of the Hall liquid had been investigated earlier by Kivelson, Lee and Zhang [2]. The authors have introduced a law of corresponding states, which allows them to construct the phase diagram and find the relation between transport coefficients, and a selection rule which governs the transition between pairs of QH states. Briefly, the electrons are represented as bosons carrying an odd integer number k of flux quanta; as a result, at filling factors $\nu = 1/k$ the effective magnetic field seen by bosons is zero, which allows them to condense into a superfluid state. These are the stable plateaux of the FQHE, characterized by conductivities $\sigma_{xx} = 0$ and $\sigma_{xy} = e^2/h\nu$. Changing the filling factor results in a non-zero effective magnetic field which produces vortices; at first, these are trapped by disorder, but at a certain critical density, corresponding to the transition point, they become delocalized. Further away, vortices will condense into a superfluid and the original bosons will be localized: the system becomes insulating. Thus, vortices and bosons are, even at this very qualitative level of description, dual to each other. These ideas form the basis for a more quantitative explanation of Shahar's experiment [1, 3].

An important issue [2] is the issue of superuniversality of the transition between Hall states. It was known from previous experimental and theoretical work [4] that the correlation length exponent is the same for all the transitions (i.e., plateau independent) and, moreover, equal to 1 for noninteracting electrons in the lowest Landau level. Recent experimental data also indicate that indeed the plateau–plateau and plateau–insulator transitions are in the same universality class [5].

In their theoretical investigation, Kivelson, Lee and Zhang argued that the transitions are indeed superuniversal by analysing the critical conductivities as given by the linear response theory combined with the duality properties of the system. This analysis turned out to be connected with the concept of modular invariance introduced by Lütken and Ross for fractional quantum Hall states [6] and was later refined by Fradkin and Kivelson [7]. The connection between particle–vortex duality, *SL* (2,*Z*) symmetry, the correspondence laws and electromagnetic response functions has been further explored by several authors in the linear response regime and beyond [8].

The Fradkin–Kivelson models [7] have a manifest modular symmetry in the response functions, which makes them suitable for the study of quantum critical behaviour at zero temperature in a variety of two-dimensional systems. Many issues can be addressed, the most interesting ones being related to how the loop–loop correlation functions (and hence the conductivities) behave around the fixed modular points (which are assimilated to the quantum critical points).

Despite these advantages there remains an important problem that was omitted from the analysis [7]: it was assumed that the particle–hole symmetry is exact and that the average magnetic field is zero. However, if one wants to describe a real condensed-matter system, the issue of finite densities and finite magnetic fields (finite filling factors) is crucial. In this paper, we will investigate this question and show that it can be addressed by introducing topological loops (loops that wind around one of the directions—typically the time) both for the currents and for the magnetic fields. These will define filling factors and dual filling factors which will have themselves special symmetry properties under the modular group *SL* (2,*Z*).

2. Topological currents

The models we will discuss [7] are constructed on a (2+1)-dimensional cubic lattice (the third dimension is time) with periodic boundary conditions. The worldlines of the particles (they can also be quasiparticle excitations relative to a certain ground state) are represented by an integer-valued vector field $\{l_\mu(\vec{r})\}$, where $\mu = 1, 2, 3$ gives the direction and $\vec{r}$ indexes the site of the lattice. The vectors $l_\mu(\vec{r})$ are conserved, that is $\partial_\mu l_\mu(\vec{r}) = 0$, which has as a general solution

$$l_\mu(\vec{r}) = \epsilon_{\mu\nu\lambda}\partial_\nu\phi_\lambda(\vec{r}) + \bar{l}_\mu(\vec{r}). \qquad (1)$$

In lattice gauge theories, the vectors $\phi_\lambda(\vec{r})$ should be defined on the so-called 'dual' lattice; however, for most of what follows we need not distinguish, notationally or otherwise, between the two lattices or between the differential lattice derivative and the differential operator in the continuum (strictly speaking, $\partial_\mu(\vec{r})$ should be replaced by $\Delta_\mu(\vec{r})$, the lattice derivative, with a Fourier transform $\Delta_\mu(\vec{k}) = -\mathrm{i}(\exp \mathrm{i}\, k_\mu - 1)$). The worldlines $\bar{l}_\mu(\vec{r})$ are topological currents: they wind around one of the dimensions, e.g., the time dimension, in which case they produce a finite density. They cannot be written as the lattice curl of a field with periodic boundary conditions.

Similarly, a constant magnetic field perpendicular to the two-dimensional electron gas would exist in the third spatial dimension; it can also be formally written as

$$B_\mu(\vec{r}) = B\delta_{\mu,3} = \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda(\vec{r}), \tag{2}$$

that is, a topological line along the time direction at each point in the lattice (δ is the Kronecker symbol). The vector potential A resides in the $\mu = 1, 2$ plane.

Consider first the situation [7] in which there are no topological currents. The statistics of the loops l_μ are implemented in the action by a kernel $K_{\mu\nu}$, whose Fourier transform is $K_{\mu\nu}(\vec{k}) = 2\pi f C_{\mu\nu}(\vec{k})/\sqrt{k^2}$, and $C_{\mu\nu}(\vec{k}) = \mathrm{i}\epsilon_{\mu\nu\lambda}k_\lambda/\sqrt{k^2}$ is the Chern–Simons tensor. This comes from the fact that, given two loops, their linking number is $\sum_{\vec{r},\vec{r}'} l_\mu(\vec{r})C_{\mu\nu}(\vec{r}-\vec{r}')l_\nu(\vec{r})/\sqrt{-\partial^2}$. Thus, if the action has a Chern–Simons term of the form $\sum l_\mu K_{\mu\nu} l_\nu$, it will be periodic under the transformation $f \to f + 1$. These relations can be checked if one considers 'elementary loops' on the lattice (loops which are the size of the smallest squares of the lattice) which are obtained from the elementary dual lattice vector $\phi^{(e)}_\mu(\vec{r}) = \delta_{\mu,\mu_0}\delta_{\vec{r},\vec{r}_0}$: then $l_\mu(\vec{r}) = \epsilon_{\mu\nu\lambda}\partial_\nu\phi^{(e)}_\sigma(\vec{r})$ are the smallest loops, centred at $\vec{r}_0$, and circling the vector $\delta_{\mu,\mu_0}\delta_{\vec{r},\vec{r}_0}$. Any other (larger) loop can be constructed by concatenating elementary loops.

Let us now see what happens with the topological terms $\bar{l}_\mu(\vec{r})$. First we note that any configuration with a certain average density ρ (ρ is a natural number) can be constructed from ρ topological loops in the form of straight lines along the time direction $\delta_{\mu,3}\delta_{r_x,0}\delta_{r_y,0}$, where $\vec{r} = (r_x, r_y, r_z)$ by adding (concatenating) elementary loops: any possible configuration with density ρ can be obtained. Thus, when writing the action it will be enough to sum over elementary loops. In order to describe the system properly, statistics must be again considered: but formally, nothing needs to be changed in the action, since now the term $\sum_{\vec{r},\vec{r}'} l_\mu(\vec{r})C_{\mu\nu}(\vec{r}-\vec{r}')l_\nu(\vec{r})/\sqrt{-\partial^2}$ will give precisely the braiding number of the currents.

We are now half way through the argument. Formally, separating the topological and the loop part of the currents does not help much: we would prefer a unified description for both, since there is no clear way of how to make such a separation in the dual picture. To find this representation it is instructive to have a look at the physics we want to describe: a two-dimensional sample in a uniform magnetic field. In reality, what we have is a finite sample, observed in a finite amount of time, and penetrated by magnetic field lines which are not uniform and infinite, but they close somewhere (ideally far away from the sample). Our model is indeed an idealization of the real situation: let us then restore also in the theory the true experimental situation, and 'enlarge' the space onto which our action is defined. In this way, we will choose to continue any topological current outside the cube defined by the sample (both in space and time) into a loop. Similarly, we will take any line of magnetic field and close it into a loop outside the sample. We will denote the quantities defined in the 'enlarged' space with a tilde sign. Any current loop can be generated, in the enlarged space, from a field $\tilde{\phi}$

$$\tilde{l}_\mu(\vec{r}) = \epsilon_{\mu\nu\lambda}\partial_\nu\tilde{\phi}_\lambda(\vec{r}), \tag{3}$$

and this expression includes now the topological worldlines. The price we pay is that the density on the sample is no longer fixed: some loop fluctuations which intersect only once the sample will add particles to the sample, others will add holes (subtract particles). It will be the magnetic field which will determine, in the end, which configurations are more favourable (for zero magnetic field, loops with opposite circulation have equal weight, so the effect is null on average). The situation is not without precedent. Our procedure is analogous to going from the canonical to the grand-canonical description of a system, or from number-conserving to $U(1)$ broken gauge symmetry Hamiltonians.

The action we will consider corresponds to Fradkin–Kivelson model II [7], which is an explicitly self-dual effective theory of charges with fractional statistics living on a plane, with long-range retarded interactions due to electromagnetic fluctuations in 3+1 dimensions. In the enlarged space this reads

$$S[\tilde{l}] = \frac{1}{2}\sum \tilde{l}_\mu (G_{\mu\nu} + \mathrm{i}K_{\mu\nu})\tilde{l}_\nu + \mathrm{i}e\sum \tilde{l}_\mu \tilde{A}_\mu + \sum \frac{h}{\sqrt{-\partial^2}}\tilde{l}_\mu \tilde{B}_\mu. \tag{4}$$

In this expression, the kernel $G_{\mu\nu}$ is defined by

$$G_{\mu\nu}(\vec{k}) = \frac{2\pi g}{\sqrt{k^2}} T_{\mu\nu}(\vec{k}), \tag{5}$$

where the tensor $T_{\mu\nu}(\vec{k}) = \delta_{\mu\nu} - k_\mu k_\nu / k^2$. Thus, the corresponding term defines a long-range repulsive matter interaction. $G_{\mu\nu}$ has the form of an electromagnetic propagator, and it can be understood as describing essentially retarded nonlocal Coulomb-type interactions between particles. The statistics is given by the Chern–Simons term, as explained above. The strength of the coupling between the currents and the external vector potential is set by the electrical charge e. The chemical potential is introduced through the 'time component' of the field $\tilde{A}$; indeed, we note that in the action the term eA_3 is multiplied by the number of particles. Therefore, in a typical Hall experiment we will take the magnetic field $\tilde{B}$ orthogonal to the two-dimensional sample; then the vector field A which generates this potential will have components only on the sample plane. A_3 can be then chosen independently. We will see in the following that this methodology leads to finite particle densities (filling factors) in the sample area. Finally, the last term couples long-distance currents with the magnetic field and can be regarded as a background 'charge' for the long-range matter interaction. Indeed, up to constant terms, the last term can be included in the matter part as

$$\frac{1}{2}\sum \left(\tilde{l}_\mu + \frac{h}{2\pi g}\tilde{B}_\mu\right) G_{\mu\nu} \left(\tilde{l}_\nu + \frac{h}{2\pi g}\tilde{B}_\nu\right), \tag{6}$$

which reminds of the form of these matter terms in the standard effective theory of QHE [9].

3. Duality transformation

We start with the general expression for the action (4) to which we add a probe field $\tilde{a}_\mu(\vec{r})$ coupled linearly with the worldlines; this will serve us to find the polarizations and the conductivities. The action is

$$S[\tilde{l}; \tilde{a}] = S[\tilde{l}] + \sum \tilde{l}_\mu \tilde{a}_\mu \tag{7}$$

and the partition function, with the use of Poisson summation formula, becomes

$$\mathcal{Z}[\tilde{a}] = \sum_{\tilde{l}_\mu} \delta(\partial_\mu \tilde{l}_\mu)\, \mathrm{e}^{-S[\tilde{l};\tilde{a}]} \tag{8}$$

$$= \sum_{m_\mu} \int \mathcal{D}\tilde{l}_\mu \delta(\partial_\mu \tilde{l}_\mu) \exp\left(-S[\tilde{l};\tilde{a}] + 2\pi\,\mathrm{i} \sum \tilde{m}_\mu \tilde{l}_\mu\right). \tag{9}$$

Here we have introduced a new integer-valued field $\tilde{m}_\mu(\vec{r})$, defined also on the whole extended space, which will allow us to define the dual field $\tilde{L}_\mu(\vec{r})$ by

$$\tilde{L}_\mu(\vec{r}) = \epsilon_{\mu\nu\lambda}\partial_\nu \tilde{m}_\lambda(\vec{r}). \tag{10}$$

The dual form of the partition function results in the form

$$\mathcal{Z}^d[\tilde{a}] = \mathcal{Z}_0 \sum_{\tilde{L}_\mu} \delta(\partial_\mu \tilde{L}_\mu)\, \mathrm{e}^{-S^d[\tilde{l};\tilde{a}]}. \tag{11}$$

To obtain S^d we follow the standard procedure of gauge fixing and integration of the field $\tilde{\phi}$. We obtain, up to constant terms,

$$\begin{aligned} S^d[\tilde{L};\tilde{a}] = \frac{1}{2}\sum \tilde{L}_\mu\big(G^d_{\mu\nu} + \mathrm{i}K^d_{\mu\nu}\big)\tilde{L}_\nu + \mathrm{i}e^d \sum \tilde{L}_\mu \tilde{A}_\mu + h^d \sum \frac{1}{\sqrt{-\partial^2}} \tilde{L}_\mu \tilde{B}_\mu \\ - \mathrm{i}\sum \tilde{L}(g^d C_{\mu\nu} + \mathrm{i} f^d T_{\mu\nu})\tilde{a}_\nu - \frac{e}{2\pi}\sum \tilde{a}_\mu (g^d T_{\mu\nu} + \mathrm{i} f^d C_{\mu\nu}) \\ \times \big(\mathrm{i}e\sqrt{-\partial^2}\tilde{A}_\nu + hT_{\nu\lambda}\tilde{B}_\lambda\big) + \frac{1}{2}\sum \tilde{a}_\mu \frac{\partial^2}{(2\pi)^2}\big(G^d_{\mu\nu} + \mathrm{i}K^d_{\mu\nu}\big)\tilde{a}_\nu. \end{aligned} \tag{12}$$

In this expression, the dual quantities are given by

$$g^d = \frac{g}{f^2 + g^2}, \tag{13}$$

$$f^d = -\frac{f}{f^2 + g^2}, \tag{14}$$

$$e^d = ef^d + fg^d, \tag{15}$$

$$h^d = -eg^d + hf^d. \tag{16}$$

Using $\tilde{a}_\mu$ as a probe field (as always in Gaussian theories, $\langle \tilde{a}_\mu \rangle = 0$) we can calculate the correlation functions (up to any order). The loop–loop correlation function is the most interesting one.

4. The loop–loop correlation function

The loop–loop correlation function is found as

$$\langle \tilde{l}_\mu \tilde{l}_\nu \rangle = \frac{\delta^2}{\delta\tilde{a}_\mu \delta\tilde{a}_\nu} \ln \mathcal{Z}[\tilde{a}]|_{\tilde{a}=0}$$

$$= \frac{\sqrt{-\partial^2}}{2\pi}(d_S T_{\mu\nu} - \mathrm{i} d_A C_{\mu\nu}), \tag{17}$$

where we have introduced symmetric and antisymmetric rescaled amplitudes d_A and d_S [7]. The calculation methodology is the same as in the absence of topological loops [7]; below we just recapitulate the results which, not surprisingly, will be the same in the 'extended' theory.

Indeed, now we calculate the correlations of a Gaussian form: these are left unchanged if we couple linearly finite terms (the magnetic field and its vector potential). With the notation

$$z = f + \mathrm{i}g, \tag{18}$$

$$H = h - \mathrm{i}e, \tag{19}$$

$$D(z) = d_A(z) - \mathrm{i}d_S(z), \tag{20}$$

$$\begin{aligned}\Sigma(z; H) &= \sigma_{xy}(z; H) + \mathrm{i}\sigma_{xx}(z; H)\\ &= \Sigma_0 + \frac{H^2}{2\pi} D^*(z),\end{aligned} \tag{21}$$

we obtain, by putting the condition of self-duality (same conductivities at z and $z^d = f^d + \mathrm{i}g^d = -1/z$), that

$$D\left(-\frac{1}{z}\right) = z + z^2 D(z). \tag{22}$$

This has to be completed with the periodicity condition $D(z) = D(z+1)$. The consequences of these relations, which suggest an *SL* (2,*Z*) modular invariance symmetry, are discussed in [7]. At some fixed modular points, the conductivities can be evaluated and universal constant values are found for bosons and fermions. This is consistent with the arguments for superuniversality [2].

Now, when looking back from the enlarged space to the sample space, we see that this amounts only to a restriction in the set of values taken by $\vec{r}$; but the relations between correlations and conductivities, etc remain the same. Thus, the introduction of topological terms preserves the symmetries of the transport coefficients (duality and periodicity).

5. Finite filling fraction

We now analyse the issue of finite filling factors (the role of topological terms). As noted before, working in the enlarged space is analogous to the non-conservation of the number of particles. The only way to say something about the number of particles is through averages on all the configurations, with weights given by the exponentiated action. We obtain

$$\langle \tilde{l}_\mu \rangle = \frac{\delta}{\delta \tilde{a}_\mu} \ln \mathcal{Z}[\tilde{a}]. \tag{23}$$

With the use of the duality transformation (12), we find, after some algebra, the following relation between the average current and its dual

$$\langle \tilde{l}_\mu \rangle = (f^d T_{\mu\nu} - \mathrm{i}g^d C_{\mu\nu}) \langle \tilde{L}_\nu \rangle - \frac{1}{2\pi}(e^d T_{\mu\nu} + \mathrm{i}h^d C_{\mu\nu}) \tilde{B}_\nu. \tag{24}$$

To solve this equation, it is enough to note that the equation is linear, and that it can be split into a symmetric and an asymmetric part. The general form of the solution can be taken as

$$\langle \tilde{l}_\mu \rangle = \frac{1}{2\pi}(\tilde{u} T_{\mu\nu} - \mathrm{i}\tilde{v} C_{\mu\nu}) \tilde{B}_\nu. \tag{25}$$

Now we consider the average currents only on the sample space. The equation above confirms that imposing a topological magnetic field in the sample space has as a consequence the appearance of finite topological currents which turn out to have interesting symmetry properties, as we will see below. The above formula is of course general, valid on the full enlarged space; but the physically relevant quantities (such as the flux of magnetic field or the total density) have to be defined on the sample space. Thus equation (25), if integrated for instance over a constant-time plane of the sample, gives a relation between the magnetic flux and the density of particles: clearly, u can be interpreted as a filling factor, while v can be understood as a vortex component, describing the relation between $\tilde{\phi}(\vec{r})$ and $\tilde{A}(\vec{r})$. Typically, for Hall systems in this region A_μ for $\mu = 1, 2$ are such that they generate a constant magnetic field perpendicular to the sample, while eA_3 is the chemical potential; the component $\langle \tilde{l}_3 \rangle$ will then contain not only a contribution proportional to u, as in the standard field theory of FQHE, but also a part which will be given by the chemical potential. This extra term, which has the meaning of a background charge density, is the price we have to pay for having a self-dual theory. This situation is best characterized mathematically by working in general with complex filling factors.

We now want to investigate the properties of u and v as functions of $z = f + \mathrm{i}g$. Let us introduce the function

$$\chi(z; H) = v(z; H) - \mathrm{i}u(z; H), \tag{26}$$

which, as noted above, has the meaning of a complex filling factor, and plug it into equation (24). We get

$$\chi\left(-\frac{1}{z}; -\frac{H}{z^*}\right) = -H^* - z\chi(z; H). \tag{27}$$

This shows that the H-dependence of χ is trivial, and a function which depends only on z is

$$\Phi(z) = \frac{\chi(z; H)}{H^*}. \tag{28}$$

Equation (27) then implies the following duality relation for $\Phi(z)$:

$$\Phi\left(-\frac{1}{z}\right) = z + z^2\Phi(z), \tag{29}$$

and obviously the periodicity holds as well,

$$\Phi(z + 1) = \Phi(z). \tag{30}$$

These properties allow us to identify $\Phi(z) \equiv D(z) = d_A(z) - \mathrm{i}d_S(z)$. To recapitulate, we obtained $\chi = H^*D$, or

$$u(z; e, h) = -ed_A(z) - hd_S(z), \tag{31}$$

$$v(z; e, h) = hd_A(z) + ed_S(z). \tag{32}$$

Now, using equation (21) we can write the following relation between the complex conductivity Σ and the complex filling factor χ,

$$\Sigma(z; H) = \Sigma_0 + \frac{1}{2\pi}H\chi(z; H), \tag{33}$$

or, in a detailed form,

$$\sigma_{xy} = \sigma_{xy}^0 + \frac{1}{2\pi}(hv - eu), \tag{34}$$

$$\sigma_{xx} = \sigma_{xx}^0 - \frac{1}{2\pi}(uh + ev). \tag{35}$$

For $h = v = 0$, which is the case most analogous to the standard treatment [9] of the FQH effect, we obtain, ignoring the background conductivities, $\sigma_{xx} = 0, \sigma_{xy} = -eu/2\pi$, where now u is the filling factor.

In an effective theory of FQHE in *continuum* this relation can be easily obtained by writing the classical equations of motion for the fields (that is, there is no need for duality transformations). In our formulation we would obtain [10] that the action

$$S^{(\mathrm{cont})}[\tilde{l}] = \mathrm{i}\pi f \sum \tilde{l}_{\mu} \frac{1}{\sqrt{-\partial^2}} C_{\mu\nu} \tilde{l}_{\nu} + \mathrm{i}e \sum \tilde{l}_{\mu} \tilde{A}_{\mu} \tag{36}$$

yields, when taking the saddle-point condition $\delta/\delta\tilde{\phi}_{\mu} S^{(\mathrm{cont})}[\tilde{l}] = 0$, that

$$\tilde{l}_{\mu} = \frac{e}{2\pi} \frac{1}{f} \epsilon_{\mu\nu\lambda} \partial_{\nu} A_{\lambda}, \tag{37}$$

which puts in evidence $1/f$ as being both the filling factor and conductivity; this effective theory describes a Laughlin states of weight f [11].

In principle, one can apply the same strategy to our *discrete* system. The equation of motion for the classical field $\tilde{l}$ has to be the same as the average of the quantum field (which just introduces fluctuations around the mean field value). The classical equation of motion is obtained from (9) by $\delta/\delta\tilde{\phi}_{\mu}(S[\tilde{l}] - 2\pi\mathrm{i}\sum \tilde{l}_{\mu}\tilde{m}_{\mu}) = 0$. The difference is that in order to take the functional derivative of the action we need a continuous $\tilde{\phi}$, so the equation for $\tilde{l}_{\mu}$ will contain not only the electromagnetic fields (as in the continuum case), but also the dual field $\tilde{L}$. We get

$$\tilde{L}_{\mu} = (fT_{\mu\nu} - \mathrm{i}gC_{\mu\nu})\tilde{l}_{\nu} - \frac{1}{2\pi}(eT_{\mu\nu} + \mathrm{i}hC_{\mu\nu})\tilde{B}_{\nu} \tag{38}$$

which is exactly the dual of (24) with classical fields instead of averages. This checks the consistency of our approach.

The modular invariance of the problem allows us to determine the values of the filling factors at the modular fixed points which are the critical points of the theory. At the bosonic point $z_0^{(b)} = \mathrm{i}$ we have $d_A(z_0^{(b)}) = 0$ and $d_S(z_0^{(b)}) = -1/2$, yielding

$$u(z_0^{(b)}) = \frac{h}{2}, \tag{39}$$

$$v(z_0^{(b)}) = -\frac{e}{2}, \tag{40}$$

$$\sigma_{xx}(z_0^{(b)}) = \sigma_{xx}^0 + \frac{1}{4\pi}(e^2 - h^2), \tag{41}$$

$$\sigma_{xy}(z_0^{(b)}) = \sigma_{xy}^0 + \frac{1}{2\pi}eh. \tag{42}$$

At the fermionic fixed point $z_0^{(f)} = 1/2 + \mathrm{i}\sqrt{3}/2$, $d_A(z_0^{(f)}) = 0$ and $d_S(z_0^{(f)}) = -\sqrt{3}/3$. Then

$$u(z_0^{(f)}) = \frac{h\sqrt{3}}{3}, \tag{43}$$

$$v(z_0^{(f)}) = -\frac{e\sqrt{3}}{3}, \tag{44}$$

$$\sigma_{xx}(z_0^{(f)}) = \sigma_{xx}^0 + \frac{\sqrt{3}}{6\pi}(e^2 + h^2), \tag{45}$$

$$\sigma_{xy}(z_0^{(f)}) = \sigma_{xy}^0 + \frac{\sqrt{3}}{3\pi}eh. \tag{46}$$

These values can be now determined at any other fixed points, by applying the modular transformation. Not only the conductivities calculated at these points are universal, but also the complex filling factor χ.

6. Hierarchy of states

The model we have analysed has a rich topological order, which can be further explored by constructing a hierarchy of states, in analogy with FQH liquids [12]. Let us consider the case $g = h = 0$ and f odd (fermions). In this case, the duality transformation results in $f^d = -1/f$, $e^d = -e/f$, and $h^d = g^d = 0$, corresponding to the group $SL\,(2,Z)$ restricted on the real axis. The action

$$S[\tilde{l}|g = h = 0] = \frac{\mathrm{i}}{2}\sum \tilde{l}_\mu K_{\mu\nu}\tilde{l}_\nu + \mathrm{i}e\sum \tilde{l}_\mu \tilde{A}_\mu, \tag{47}$$

is similar to the effective action for the $1/f$ FQH electrons, with the difference that the fields are discrete. After using Poisson's formula, the l become continuous, but the price we pay is a new term in the action, $S[\tilde{l}|g = h = 0] \to S[\tilde{l}|g = h = 0] - 2\pi\mathrm{i}\sum \tilde{m}_\mu \tilde{l}_\mu$. From the point of view of the standard effective theory of the FQH effect, this extra term describes an excitation on the $1/f$ state of the electron liquid. Indeed, the ground state of the same system on a grid cannot be the same as that in continuum, for the simple reason that this would imply a filling factor of $1/f$, but then, for, say, one unit of flux, one is not able to find any $\langle\tilde{l}_0\rangle$, since these fields are integers. The system then responds to this restriction by creating the quasiparticles $\tilde{m}$. As a result, the filling factor is also altered from the value $1/f$ to a new one, fixed by the modular symmetry of the system and reflected in equations (31), (32). This state could be called a $D(f)$-state, in analogy to its continuous version, the $1/f$ Laughlin state.

The next question to ask is what happens when one excites a $D(f)$-state. Let us index the initial system with the subscript 1 and the quasiparticles with 2. Then, the quasiparticle current $\tilde{l}_{2,\mu}$ will couple with the action of the $D(f)$-state through a linear term in the gauge field $\tilde{\phi}_{1,\mu}$, namely $2\pi\sum \tilde{l}_{2,\mu}\tilde{\phi}_{1,\mu}$. The action now looks like

$$S[\tilde{l}_1|g = h = 0] - 2\pi\mathrm{i}\sum \tilde{m}_{1,\mu}\tilde{l}_{1,\mu} + 2\pi\sum \tilde{l}_{2,\mu}\tilde{\phi}_{1,\mu} + \cdots. \tag{48}$$

To find out the rest, we use a standard argument [10]: if we do not allow the field $\tilde{\phi}_{1,\mu}$ to respond to the insertion of $\tilde{l}_{2,\mu} = \epsilon_{\mu\nu\lambda}\partial_\nu\tilde{\phi}_{2,\lambda}$, the quasiparticles will effectively move as if they were in a magnetic field $\tilde{l}_{1,\mu}$. When their filling factor is $1/f_2$ they will be in a Laughlin ground state. It is straightforward to show [10, 13], that these quasiparticles are bosons, thus f_2 is even, and to complete the action with the corresponding Chern–Simons term. The final expression, corresponding to the second level of the hierarchy, is then

$$2\pi\mathrm{i}\left\{\frac{f_1}{2}\sum \tilde{\phi}_{1,\mu}\partial_\nu\tilde{\phi}_{1,\lambda}\epsilon_{\mu\nu\lambda} + \sum\left(\frac{e}{2\pi}\tilde{B}_\mu - \tilde{L}_{1,\mu}\right)\tilde{\phi}_{1,\mu}\right.$$
$$\left. + \frac{f_2}{2}\sum \tilde{\phi}_{2,\mu}\partial_\nu\tilde{\phi}_{2,\lambda}\epsilon_{\mu\nu\lambda} + \tilde{\phi}_{1,\mu}\partial_\nu\tilde{\phi}_{2,\lambda}\epsilon_{\mu\nu\lambda}\right\}. \tag{49}$$

One can now either write the equations of motion with respect to the fields $\tilde{\phi}_{1,\mu}$ and $\tilde{\phi}_{2,\mu}$ as in [10], or, equivalently, integrate out the fields $\tilde{\phi}_{2,\mu}$. The later procedure gives for the resulting action

$$2\pi\mathrm{i}\left\{\frac{1}{2}\left(f_1 - \frac{1}{f_2}\right)\sum \tilde{\phi}_{1,\mu}\partial_\nu\tilde{\phi}_{1,\lambda}\epsilon_{\mu\nu\lambda} + \sum\left(\frac{e}{2\pi}\tilde{B}_\mu - \tilde{L}_{1,\mu}\right)\tilde{\phi}_{1,\mu}\right\}. \tag{50}$$

Thus, the statistics changes as $f \to f_1 - 1/f_2$, and the second level hierarchy is a $D(f_1 - 1/f_2)$-state. The method can be extended to the third, fourth, and so on level, with the result that we get, for f_1 odd and f_n, $n > 1$, even, a state

$$D(f) \to D\left(f_1 - \cfrac{1}{f_2 - \cfrac{1}{\cdots - \frac{1}{f_n}}}\right). \tag{51}$$

This relation shows how the symmetries of the model are reflected not only in the ground state but also in the structure of the excitations. We have here a hierarchical structure analogous to that of FQH liquids.

7. Conclusions

We have described how to introduce topological terms in a lattice gauge theory of matter with Chern–Simons terms. We showed that the symmetry of the complex filling factor is the same as that of the conductivities, namely can be obtained by the action of 'duality' ($z \to -1/z$) and 'periodicity' ($z \to z + 1$), which suggests a modular invariance at work. The theory obtained serves as a generic model for systems which exhibit self-dual properties at phase transitions.

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References

[1] Shahar D, Tsui D C, Shayegan M, Shimshoni E and Sondhi S L 1996 *Science* **274** 589
[2] Kivelson S, Lee D H and Zhang S C 1992 *Phys. Rev.* B **46** 2223
[3] Shimshoni E, Sondhi S L and Shahar D 1997 *Phys. Rev.* B **55** 13730
[4] Wei H P, Tsui D C, Paalanen M A and Pruisken A M M 1989 *Phys. Rev. Lett.* **61** 1294
Wong L W, Jiang H W, Trivedi N and Palm E 1995 *Phys. Rev.* B **51** 18033
see also Sondhi S L, Girvin S M, Carini J P and Shahar D 1997 *Rev. Mod. Phys.* **69** 315
Huckestein B 1997 *Rev. Mod. Phys.* **69** 315
[5] van Schaijik R T F, de Visser A, Olsthoorn S M, Wei H P and Pruisken A M M 2000 *Phys. Rev. Lett.* **84** 1567
[6] Lütken C A and Ross G C 1993 *Phys. Rev.* B **48** 2500
Lütken C A and Ross G C 1993 *Phys. Rev.* B **48** 11837
Lütken C A 1993 *Nucl. Phys.* B **396** 670
[7] Fradkin E and Kivelson S 1996 *Nucl. Phys.* B **474** 543
[8] Pryadko L P and Zhang S -C 1996 *Phys. Rev.* B **54** 4953
Pryadko L P 1997 *Phys. Rev.* B **56** 6810
Burgess C P and Dolan B P 2001 *Phys. Rev.* B **63** 155309
Burgess C P and Dolan B P 2002 *Phys. Rev.* B **63** 155323
[9] Girvin S M and MacDonald A H 1987 *Phys. Rev. Lett.* **58** 1252
Zhang S C, Hansson T H and Kivelson S 1989 *Phys. Rev. Lett.* **62** 82
Read N 1989 *Phys. Rev. Lett.* **62** 86
Lopez A and Fradkin E 1991 *Phys. Rev.* B **44** 5246
Lopez A and Fradkin E 1993 *Phys. Rev.* B **47** 7080
[10] Wen X -G 1995 *Adv. Phys.* **44** 405
[11] Laughlin R B 1983 *Phys. Rev. Lett.* **50** 1395
[12] Haldane F D M 1983 *Phys. Rev. Lett.* B **51** 605
Halperin B I 1984 *Phys. Rev. Lett.* **52** 1583
Girvin S 1984 *Phys. Rev.* B **29** 6012
MacDonald A H and Murray B D 1985 *Phys. Rev.* B **32** 2707
Fisher M P A and Lee D H 1989 *Phys. Rev. Lett.* **63** 903
Jain J K 1989 *Phys. Rev. Lett.* **63** 199
[13] Fradkin E 1991 *Field Theories of Condensed Matter Systems* (Redwood City, CA: Addison-Wesley)